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Repairing broken algebras

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Abstract: We consider theories characterized by a set of Ward operators which do not form a closed algebra. We impose the Slavnov–Taylor identity built out of the Ward operators and we derive the acceptable breaking of the algebra and the general form of the classical action. The 1PI generating functional is expressed in terms of the known quantities characterizing the theory and of a nontrivial integrability condition. As a nontrivial application of our formalism, we discuss the N=4 supersymmetric nonlinear sigma model.

Keywords: BRST Quantization, BRST Symmetry.

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Contents

1. Introduction

The approach to symmetries *via* the introduction of anticommuting ghosts and hence the construction of a BRS nilpotent operator has provided a powerful tool to analyze the renormalization of quantum field theories with nonlinearly realized symmetries, both local and global [1]

It was soon realized that the extension of the original symmetric action to one containing external fields coupled to the BRS variations of the quantum fields, made possible to write the BRS operator in a kind of universal way as a Slavnov–Taylor identity making in addition possible the control of the behavior under renormalization of the symmetry itself. Furthermore, as extra bonus, the bilinear terms in the sources, if the power counting and ghost number allow these insertions in the action, account for the presence of symmetries realized only on–shell, i.e. modulo the equations of motion of the fields [2].

Within this framework, the renormalization program reduces, thanks to the Quantum Action Principle [3], to an algebraic discussion of the cohomology spaces of the nilpotent BRS operator.

Our aim in this paper is that of discussing and putting into evidence the information we can obtain from a nilpotent operator: we shall therefore define a BRS operator which is only related to the covariance–and not necessarily to the invariance–properties of a classical action under a set of nonlinear transformations of the quantum fields. These nonlinear transformations are not required to be a realization of some Lie algebra either, but nonetheless satisfy algebraic relations, obtained through their commutators and/or anticommutators. A step in this directions was already taken in [4], where it was described the mechanism of reconstructing a Lie algebra from a nilpotent operator and the consequences for the cohomology spaces.

Since our goal is to discuss the most general case, we try to keep the computational complexities to a minimum, and shall therefore work only with scalar fields and global ghosts, just to illustrate our method, having in mind the action of supersymmetric non linear sigma model in two spacetime dimensions, which is explicitly discussed as an example. In the same spirit, we shall not put any power counting constraint to the external field sector of the classical action, and thus we can investigate in full generality the meaning of their contribution.

The paper is organized as follows.

In section 2, we shall obtain some algebraic relations, which will leads us to the most general form of the broken algebra and of the classical action compatible with the Slavnov–Taylor identity. Moreover, we shall find a condition, derived from the external field sector of the action, whose interpretation is not straightforward, but which is verified in all the examples known in the Literature. Particularly relevant is the case of $D = 2$, $N = 4$ supersymmetric nonlinear sigma model, which is analyzed in section 3.

In section 4, we shall study the conditions under which it is possible to recover a Lie algebra structure for the theory, and we shall find out several possibilities for that. We shall also see that the nilpotent operator one can construct in the generic case of transformations not related to a Lie algebra invariance of the classical action, leads to a trivial cohomological problem. When we restrict to a Lie algebra invariance, we shall recover a possible non-triviality of the cohomology spaces and we shall also find that the on–shell realization of the symmetry is indeed the most general case we can have.

Our conclusions are finally drawn in section 5.

2. Imposing the Slavnov–Taylor identity

Consider a classical action $I(\phi^a)$, built on a flat spacetime of generic dimension D and depending on a number of fields $\{\phi^a(x)\}$ — which for simplicity we assume to be scalar — labeled by an index a. Let $\{W_i\}$ be a set of Ward operators described by the functionals

$$W_i = \int d^D x \, W_i^a(\phi) \frac{\delta}{\delta \phi^a} \,, \tag{2.1}$$

where $W_i^a(\phi)$ depends on the fields in general in a nonlinear way. The Ward operators (2.1) are the functional realizations of the nonlinear field transformations

$$\delta_i \phi^a(x) = W_i^a(\phi) \,. \tag{2.2}$$

We are interested in treating the case in which the operators W_i do not form a closed algebra. The most general case is indeed given by the following structure

$$[W_i, W_j] = D_{ij} \,. \tag{2.3}$$

The expression (2.3) covers all possible algebraic structures, including the closed ones, obtained when the antisymmetric operator D_{ij} is given by

$$D_{ij} = f_{ijk} W_k \tag{2.4}$$

for some structure constants f_{ijk}.

What is more difficult to treat, is the case of open algebras. In this case indeed D_{ij} represents an obstruction. Examples of such open structures are given by the supersymmetry algebra, where the commutator between two supersymmetries finds as obstructions to the closure on traslations, the equations of motion and the gauge transformations [5]. Another relevant example of open algebras, are the reducible symmetries of Batalin–Vilkovisky [2], characterizing for instance a class of topological models [6]. In such theories, one lands on a BRS operator which is nilpotent only once the equations of motion are used.

We would like here to work on a very general ground, without referring to a particular model. we shall find the most general form of the breaking D_{ij} in (2.3), embedded in the Slavnov–Taylor identity holding for the 1PI generating functional Γ.

A convenient way to proceed is to introduce global ghosts C^i and λ in order to write a nilpotent BRS operator

$$\begin{aligned} s &= C^i W_i + \lambda \frac{1}{2} C^i C^j D_{ij} - \frac{\partial}{\partial \lambda} \\ &= W + \lambda d - \frac{\partial}{\partial \lambda} \,, \end{aligned} \tag{2.5}$$

where we defined

$$W \equiv C^i W_i \,, \qquad d \equiv \frac{1}{2} C^i C^j D_{ij} \,. \tag{2.6}$$

The nilpotency of the BRS operator s is insured by the identities $W^2 = 0$ and $[W, d] = 0$ and by the fact that the ghosts C^i and λ are anticommuting grassmannian variables, to which we assign charge $+1$ and -1 respectively, in order to have a BRS operator raising by one unit the ghost charge. The Slavnov–Taylor (ST) operator corresponding to the BRS operator (2.5) reads

$$\mathcal{S}(\Gamma^{cl}) = \int d^D x \, \frac{\delta \Gamma^{cl}}{\delta \gamma_a(x)} \frac{\delta \Gamma^{cl}}{\delta \phi^a(x)} + \frac{\partial \Gamma^{cl}}{\partial \lambda} \,. \tag{2.7}$$

In (2.7), Γ^{cl} is the tree level 1PI generating functional, whose most general form is

$$\begin{aligned} \Gamma^{cl} = I(\phi^a) + \int d^D x \Big(& \gamma_a C^i W_i^a(\phi) + \gamma_a \gamma_b C^i C^j W_{ij}^{ab}(\phi) + \lambda C^i A_i(\phi) + \\ & + \lambda \gamma_a C^i C^j X_{ij}^a(\phi) + \lambda \gamma_a \gamma_b C^i C^j C^k X_{ijk}^{ab}(\phi) \Big) \,. \end{aligned} \tag{2.8}$$

A few comments on the terms appearing in Γ^{cl} are in order :

1. The $\gamma_a(x)$ are external sources coupled to the nonlinear variations $W_i^a(\phi)$ of the quantum fields $\phi^a(x)$, according to the standard method of treating nonlinear symmetries;

2. For closed algebras, only the term linear in the external sources appears in Γ^{cl}. $W_{ij}^{ab}(\phi)$, $A_i(\phi)$, $X_{ij}^a(\phi)$, and $X_{ijk}^{ab}(\phi)$ are generic polynomials in the fields $\phi^a(x)$, constrained only by the fulfillment of the ST identity

$$\mathcal{S}(\Gamma^{cl}) = 0; \tag{2.9}$$

3. We demand the conservation of the ghost charge. In order to have Γ^{cl} uncharged, the sources $\gamma_a(x)$ must be assigned charge -1;

4. Since we are considering arbitrary spacetime dimensions and we are not referring to a particular model, we do not impose any power counting restriction on Γ^{cl}. Nonetheless, we restrict ourselves to terms at most quadratic in the external sources, to keep contact with the cases of most concern. Our analysis can be trivially extended to the case of higher powers in the γ_a's.

Introducing the notation

$$\begin{aligned} C^i C^j W_{ij}^{ab}(\phi) &\equiv W^{ab}(C,\phi) \\ C^i A_i(\phi) &\equiv A(C,\phi) \\ C^i C^j X_{ij}^a(\phi) &\equiv X^a(C,\phi) \\ C^i C^j C^k X_{ijk}^{ab}(\phi) &\equiv X^{ab}(C,\phi) \end{aligned} \tag{2.10}$$

the action Γ^{cl} can be written in a more compact way :

$$\Gamma^{cl} = I(\phi^a) + \int d^Dx \left(\gamma_a W^a + \gamma_a\gamma_b W^{ab} + \lambda C^i A_i + \lambda\gamma_a X^a + \lambda\gamma_a\gamma_b X^{ab} \right) \tag{2.11}$$

It is useful to summarize in a table the ghost charges ($\Phi\Pi$) involved :

	ϕ^a	γ_a	λ	W^a	W^{ab}	A	X^a	X^{ab}	d
$\Phi\Pi$	0	-1	-1	1	2	1	2	3	2

Our approach is to require that the action Γ^{cl} (2.11) satisfies the ST identity (2.9). In this way we shall find the most general form of the breaking d (2.6) and of the functions $W^{ab}(C,\phi)$, $A(C,\phi)$, $X^a(C,\phi)$ and $X^{ab}(C,\phi)$, which for the moment are left generic. Recall that, the only known elements are the variations of the fields $W_a(\phi)$ and the classical action $I(\phi)$.

The calculation, although rather lengthy, is straightforward. It is convenient to analyze the ST identity according to the powers of $\gamma_a(x)$ and λ.

At the zero order, we find that the ST identity (2.9) is satisfied provided that

$$\int d^Dx \left(W^a \frac{\delta I}{\delta \phi^a} + A \right) = 0 \tag{2.12}$$

as it to say that the functional $A_i(\phi)$ in (2.8) can be interpreted as a breaking term. In other words, in our formalism we do not ask that the Ward operators W_i describe symmetries of the classical action $I(\phi)$, but we allow for a (in general nonlinear) breaking $A_i(\phi)$.

Considering the order $O(\gamma)$, we find the expression for $X^a(C,\phi)$:

$$X^a(C,\phi) = 2W^{ab}(C,\phi)\frac{\delta I(\phi)}{\delta \phi^b} + W^b(C,\phi)\frac{\delta W^a(C,\phi)}{\delta \phi^b} \tag{2.13}$$

in terms of the known $W^a(C,\phi)$ and the unknown $W^{ab}(C,\phi)$. Applying to both sides of (2.13) $\frac{\delta}{\delta\phi^a}$, summing over a, performing the spacetime integration and remembering that $W^2 = d$, we find

$$d = \int d^Dx \left(X^a - 2W^{ab}\frac{\delta I}{\delta \phi^b} \right) \frac{\delta}{\delta \phi^a} \,. \tag{2.14}$$

The equation (2.14) represents the most general form of d (2.6), as results from the imposition of the ST identity (2.9), written in terms of quantities appearing in the classical action Γ^{cl}. The breaking of the algebra (2.3) consists therefore of two parts, one of which is vanishing once the equations of motion are satisfied ($\frac{\delta I}{\delta\phi^a} = 0$). The other term is $\int d^Dx\, X^a \frac{\delta}{\delta \phi^a}$, and is still there even on–shell, representing the bulk of the breaking. The supersymmetry algebra [5] is an example of an algebra whose breaking has exactly the structure described in (2.14). The form of d provides moreover a simple interpretation of the terms $X^a(C,\phi)$ and $W^{ab}(C,\phi)$ appearing in (2.11). We recall that, while the functional $X^a(C,\phi)$ is determined by (2.13), $W^{ab}(C,\phi)$ is still completely unconstrained.

We go on imposing the ST identity, by selecting the term quadratic in the external fields, and we find the following expression for $X^{ab}(C,\phi)$:

$$X^{ab}(C,\phi) = W^{ac}(C,\phi)\frac{\delta W^b(C,\phi)}{\delta\phi^c} - W^{bc}(C,\phi)\frac{\delta W^a(C,\phi)}{\delta\phi^c} - W^c(C,\phi)\frac{\delta W^{ab}(C,\phi)}{\delta\phi^c} \,. \tag{2.15}$$

Finally, the ST identity at the order $O(\gamma^3)$ is satisfied provided that

$$W^{ab}(C,\phi)\frac{\delta W^{cd}(C,\phi)}{\delta\phi^a} + W^{ad}(C,\phi)\frac{\delta W^{bc}(C,\phi)}{\delta\phi^a} + W^{ac}(C,\phi)\frac{\delta W^{db}(C,\phi)}{\delta\phi^a} = 0 \,. \tag{2.16}$$

All the other constraints deriving from the ST identity are automatically satisfied once the expressions found for $A(C,\phi)$ (2.12), $X^a(C,\phi)$ (2.13), $X^{ab}(C,\phi)$ (2.15), the constraint on $W^{ab}(C,\phi)$ (2.16) and the algebraic relation $[W,d] = 0$, hold.

At this point, every term in the action Γ^{cl} (2.8) is completely determined in terms of $I(\phi)$, $W_i^a(\phi)$ and $W_{ij}^{ab}(\phi)$. Now, while the first two quantities are given, being our starting point, the functional $W_{ij}^{ab}(\phi)$ is not determined by the condition (2.16). Looking at the general expression we found for the operator d (2.14), we realize that the W^{ab}'s are related to that part of the breaking of the algebra (2.3) which is vanishing on–shell. In most explicit cases, the W^{ab}'s do not depend on the fields, and the constraint (2.16) is trivially satisfied. For a nontrivial field dependence of W^{ab}, the relation (2.16) can be interpreted as an integrability condition, whose validity is not straightforwardly guaranteed. In the next section, in fact, we discuss a case in which (2.16) has an evident geometrical meaning, not directly related to the action and its symmetries.

Once the ST identity (2.9) is satisfied, one can verify that the nilpotency of the corresponding linearized ST operator

$$\mathcal{B}_{\Gamma^{cl}} = \int d^D x \left(\frac{\delta \Gamma^{cl}}{\delta \phi^a(x)} \frac{\delta}{\delta \gamma_a(x)} + \frac{\delta \Gamma^{cl}}{\delta \gamma_a(x)} \frac{\delta}{\delta \phi^a(x)} \right) + \frac{\partial}{\partial \lambda} \tag{2.17}$$

is guaranteed.

3. An example: $D = 2$, $N = 4$ supersymmetric nonlinear sigma model

In most cases the coefficients $W^{ab}(C, \phi)$ are field independent, and hence trivially satisfy (2.16); one remarkable exception is the $D = 2$, $N = 4$ supersymmetric nonlinear sigma model, described, in absence of torsion, by the action

$$I(\phi^a) = \int d^2x\, d^2\theta\, g_{ab}(\phi)[D_+\phi^a D_-\phi^b]\,. \tag{3.1}$$

The invariant action (3.1) is written in terms of light-cone coordinates, and

$$D_\pm = \frac{\partial}{\partial \theta^\pm} + i\,\theta^\pm \frac{\partial}{\partial x^\pm}\,. \tag{3.2}$$

Moreover, depending on $N = 1$ superfields $\phi^a(x, \theta)$, the action is manifestly $N = 1$ supersymmetric. Nevertheless it has been shown [7], that the action (3.1) is invariant under the following additional three nonlinear (super)field transformations

$$\begin{aligned} \delta\phi^a &= J_{ib}^a(\phi)[\epsilon_i^+ D_+\phi^b + \epsilon_i^- D_-\phi^b] \\ &\equiv [\epsilon_i^+ W_{i+} + \epsilon_i^- W_{i-}]\phi^a \qquad i = 1, 2, 3\,, \end{aligned} \tag{3.3}$$

provided that the tensors $J_{ib}^a(\phi)$ are complex structures satisfying the SU(2) quaternionic relations

$$J_{ic}^a(\phi) J_{jb}^c(\phi) = -\delta_{ij}\delta_b^a + \epsilon_{ijk} J_{kb}^a(\phi) \tag{3.4}$$

and moreover obey

$$J_i^{ab}(\phi) = J_{ic}^{a}(\phi)g^{cb}(\phi) = -J_i^{ba}(\phi) \tag{3.5}$$

$$D_c J_{ib}^a(\phi) \equiv \partial_c J_{ib}^a(\phi) + \Gamma_{cd}^a[g] J_{ib}^d(\phi) - \Gamma_{cb}^d[g] J_{id}^a(\phi) = 0\,. \tag{3.6}$$

Equation (3.4), together with (3.5), implies that the metric is hermitian with respect to the J's, and the identity (3.6) represents the fact that the complex structures J are covariantly constant with respect to the metric ($\Gamma_{bc}^a[g]$ is the Christoffel connection).

The conditions (3.4), (3.5) are local properties which can be globally extended if and only if the complex structures J are such that the corresponding Nijenhuis tensors vanish

$$N_{iab}^c = J_{ia}^d(\partial_d J_{ib}^c - \partial_b J_{id}^c) - (a \leftrightarrow b) = 0\,. \tag{3.7}$$

Now, the condition (3.6) implies the vanishing of the Nijenhuis tensor (3.7), so that the properties (3.4) and (3.5) can be extended to the whole riemannian manifold spanned by the coordinates ϕ^a. This defines the variety to be of hyperkähler type [8]. In other words, the existence of extended supersymmetries for the action (3.1) requires that the target space is an hyperkähler riemannian manifold [7]. The quantization of this model has been performed in [9].

The supersymmetry algebra closes only on–shell, since we have

$$\{W_{i+}, W_{j-}\}\phi^a = \epsilon_{ijk} J_k^{ab} \frac{\delta I(\phi)}{\delta \phi^b}\,, \tag{3.8}$$

which is of the general type (2.3).

According to our analysis, we have to introduce in the total action terms linear in the external fields γ_a, and also terms bilinear in the external fields, which, in the notation of (2.11) and recalling the most general expression (2.14) found for the breaking of the algebra, are respectively given by

$$\begin{aligned} W^a &= J_{ib}^a(\phi)[C_i^+ D_+ \phi^b + C_i^- D_- \phi^b] \\ W^{ab} &= -\frac{1}{2}\epsilon_{ijk} J_k^{ab} C_i^+ C_j^-\,, \end{aligned} \tag{3.9}$$

where $C_i^{\pm}$ are the commuting supersymmetric Faddeev–Popov constant ghosts.

The analysis of relation (2.16) with $W^{ab}(\phi)$ specified as above requires a good amount of algebraic patience but in the end one finds that it reduces to

$$(\alpha\delta_{ij} + \alpha_{ij}) J_i^{da} \partial_d J_j^{bc} = 0\,, \tag{3.10}$$

where α and α_{ij} are constant tensors

$$\begin{aligned} \alpha &= (C_i^+ C_j^- - C_j^+ C_i^-)(C_i^+ C_j^- - C_j^+ C_i^-) \\ \alpha_{ij} &= 2\,(C_k^+ C_i^- - C_i^+ C_k^-)(C_k^+ C_j^- - C_j^+ C_k^-) \end{aligned} \tag{3.11}$$

and the left-hand side of the identity (3.10) is to be understood as antisymmetrized in the indices (a, b, c).

Hence, the complex structures $J^a_{ib}(\phi)$ must satisfy the condition

$$J^{da}_k \partial_d J^{bc}_j + J^{dc}_k \partial_d J^{ab}_j + J^{db}_k \partial_d J^{ca}_j = 0\,. \tag{3.12}$$

It is a remarkable fact that such integrability condition holds true due to the validity of (3.5) and (3.7). In conclusion, the identity (2.16), which holds true in the generic case of open algebras, when applied to the particular case of $N = 4$ supersymmetric nonlinear sigma model, shows its geometrical meaning (3.12). It is interesting to notice that the vanishing of the Nijenhuis tensor, i.e. the possibility of defining globally the complex structures, is a needed property in order to have a nilpotent operator, which therefore also carries information on the global geometrical properties of the manifold.

4. Recovering algebras

Once we have determined the most general action Γ^{cl} and the broken algebra compatible with the ST identity (2.9), the next natural step is to investigate whether it is possible to absorb the ghost λ and to find out which are the (broken) algebraic structures which allow for the absorption.

The ghost λ can be absorbed in three ways

1. through a redefinition of the ghosts C^i

$$\widehat{C}^i \equiv C^i + \lambda f^i_{jk} C^j C^k\,; \tag{4.1}$$

2. through a redefinition of the external fields

$$\widehat{\gamma}_a \equiv \gamma_a + f^b_{ai} \lambda \gamma_b C^i\,; \tag{4.2}$$

3. through a combination of (4.1) and (4.2).

Redefining the ghosts C^i as in (4.1), the action reads

$$\widehat{\Gamma}^{cl} = I + \int d^D x \left(\gamma_a \widehat{C}^i \widehat{W}^a_i + \gamma_a \gamma_b \widehat{C}^i \widehat{C}^j \widehat{W}^{ab}_{ij} \right), \tag{4.3}$$

where $\widehat{W}^a_i(\phi)$ and $\widehat{W}^{ab}_{ij}(\phi)$ are new functionals of the fields ϕ^a to be determined by means of the functionals appearing in the previous expression for Γ^{cl} (2.8). Indeed, writing back in (4.3) the ghosts C^i and identifying term by term the expressions for $\widehat{\Gamma}^{cl}$ and for Γ^{cl}, one gets

$$\begin{aligned}
W^a_i &= \widehat{W}^a_i \\
W^{ab}_{ij} &= \widehat{W}^{ab}_{ij} \\
X^a_{ij} &= -f^k_{ij} \widehat{W}^a_k \\
A_i &= 0 \\
X^{ab}_{ijk} &= f^l_{jk} \widehat{W}^{ab}_{il} + f^l_{ij} \widehat{W}^{ab}_{kl} + f^l_{ki} \widehat{W}^{ab}_{jl}\,.
\end{aligned} \tag{4.4}$$

The fact that the breaking term $A_i(\phi)$ vanishes, means that the Ward operators $\widehat{W}_i = \int d^Dx\, \widehat{W}_i^a \frac{\delta}{\delta\phi^a}$ describe symmetries of the classical action $I(\phi)$, by virtue of (2.12) Moreover, from the expression (2.13) for $X^a(C,\phi)$ we derive the algebra compatible with the absorption of the ghost λ through a redefinition of C^i:

$$\left[\widehat{W}_i, \widehat{W}_j\right] = f_{ij}^k \widehat{W}_k + 2\int d^Dx\, \widehat{W}_{ij}^{ab} \frac{\delta I}{\delta\phi^a}\frac{\delta}{\delta\phi^b}\,. \tag{4.5}$$

As it can be seen, (4.5) describes an on–shell algebra for the Ward operators $\widehat{W}_i$, whose structure constants are f_{jk}^i.

Following the same method, one finds that the ghost λ can also be absorbed through a redefinition of the external fields as in (4.2) and correspondingly the action $\widehat{\Gamma}^{cl}$ has the form (4.3) with

$$\begin{aligned}
W_i^a &= \widehat{W}_i^a \\
W_{ij}^{ab} &= \widehat{W}_{ij}^{ab} \\
X_{ij}^a &= \frac{1}{2}\left(f_{bi}^a \widehat{W}_j^b - f_{bj}^a \widehat{W}_i^b\right) \\
A_i &= 0 \\
X_{ijk}^{ab} &= f_{ci}^a \widehat{W}_{jk}^{bc}\,.
\end{aligned} \tag{4.6}$$

Again, the $\widehat{W}_i$ are symmetries of $I(\phi)$ and satisfy the algebra

$$\left[\widehat{W}_i, \widehat{W}_j\right] = \int d^Dx \left(f_{bi}^a \widehat{W}_j^b \frac{\delta}{\delta\phi^a} + 2\widehat{W}^{ab}\frac{\delta I}{\delta\phi^a}\frac{\delta}{\delta\phi^b}\right), \tag{4.7}$$

which differs from the previous obtained one (4.5), as different are the structures constants f_{bi}^a as well.

Eliminating finally the ghost λ from the theory by redefining both the ghosts C^i and the external sources γ_a, by means of (4.1) and (4.2), one still finds that the $\widehat{W}_i$ must be symmetries of the action $I(\phi)$, i.e. $A_i(\phi) = 0$, and the algebra is

$$\left[\widehat{W}_i, \widehat{W}_j\right] = f_{ij}^k \widehat{W}_k + \int d^Dx \left(f_{bi}^a \widehat{W}_j^b \frac{\delta}{\delta\phi^a} + 2\widehat{W}^{ab}\frac{\delta I}{\delta\phi^a}\frac{\delta}{\delta\phi^b}\right). \tag{4.8}$$

As one would expect, (4.8) is the direct product of the algebras (4.5) and (4.7).

We conclude this section with a remark concerning the cohomological structure of the model we are considering, with and without the constant ghost λ. In presence of λ, the cohomology of the linearized ST operator (2.17) is trivial [4], although the algebraic constraints we find are not trivial. Otherwise, when λ is not present because can be absorbed as it has been discussed, the cohomology depends on the detailed structure of the algebra one recovers. In other words, what is not trivial, is the cohomology of the ST operator constrained to its λ–independent sector. That the presence of λ somehow trivializes the cohomology of the theory can be easily seen, for instance, by using Dixon's filtration theorem [10], which states that the cohomology of a nilpotent operator s is isomorphic to a subspace of the cohomology of $s^{(0)}$, where $s^{(0)}$ is obtained by filtering s with a filtration operator $\mathcal{N}$, such that

$$s = \sum_{n \geq 0} s^{(n)} , \qquad [\mathcal{N}, s^{(n)}] = n s^{(n)} . \tag{4.9}$$

In our case, a convenient choice for $\mathcal{N}$ is

$$\mathcal{N} = \int d^D x \, \phi^a \frac{\delta}{\delta \phi^a} + C^i \frac{\partial}{\partial C^i} . \tag{4.10}$$

according to which the filtered linearized ST operator (2.17) at the lowest order is

$$\mathcal{B}^{(0)}_{\Gamma^{cl}} = \frac{\partial}{\partial \lambda} , \tag{4.11}$$

which is clearly a nilpotent operator with empty cohomology. Hence, the cohomology of the whole operator $\mathcal{B}_{\Gamma^{cl}}$ is empty, too.

5. Conclusions

We considered a theory characterized by an action $I(\phi)$ and by some nonlinear Ward operators W_i. We put ourselves in the most general case, with the W_i neither describing symmetries nor forming a closed algebra, being motivated by the fact that theories of physical relevance, like supersymmetric and topological models for instance, are of this type.

At the tree level, we found the most general 1PI generating functional Γ^{cl} embedding the action $I(\phi)$ which satisfies the Slavnov–Taylor identity (2.9) built from the W_i's and depending on the quantum fields $\phi_a(x)$, the sources $\gamma_a(x)$ and two types of global ghosts: the C^i and λ, associated to W_i and to the breaking of the algebra, respectively. Finally, everything turns out to be expressed in terms of the known quantities of the theory, i.e. the action $I(\phi)$ and the Ward operators W_i, and of the integrability condition (2.16). Our formalism has been explicitly applied to the case of $N = 4$ supersymmetric nonlinear sigma model, which shows an open algebra of the type described in this paper. In this context, the general integrability condition (2.16), and hence the Slavnov–Taylor identity which originates it, turns out to have a remarkable geometrical meaning.

In addition, by means of equation (2.14), we determined the structure of the breaking of the algebra compatible with the Slavnov–Taylor identity, and thus with the quantization of the theory.

As last step, we studied the possibility of eliminating the ghost λ, by redefining the ghosts C^i and/or the sources $\gamma_a(x)$. We found that this is possible for various types of algebras, broken only by terms vanishing on–shell, and for Ward operators W_i which, differently from the general case analyzed in this paper, describe symmetries of the theory.

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